Theory of planar curves

If M is an interval or diffeo to S^1 and if $X : M \times [0,T) \to \mathbb{R}$ is a smooth map which is an immersion $\forall\, t$, then $X(t)$ is said to a sol^n of the curve shortening flow if

$$\frac{\partial X}{\partial t}(u,t) = -\kappa(u,t)\, N(u,t)$$

κ = curvature

N = unit normal vector

Def^n:- A curve in $\mathbb{R}^n$ is a map $X : (\alpha,\beta) \to \mathbb{R}^n$ for some $-\infty \leq \alpha < \beta \leq \infty$.

e.g. a parabola $X : (-\infty,\infty) \to \mathbb{R}^2$, $X(u) = (u, u^2)$ which can be equivalently parametrised by

$X(u) = (u^3, u^6)$ or $X(u) = (2u, 4u^2)$ etc.

a circle of radius r : $X(u) = (r\cos u, r\sin u)$

w/ $[\alpha, \beta] = [0, 2\pi]$.

$\frac{dX}{du} = X'(u)$ is called the "tangent vector" at $X(u)$.

→ Curve will be immersed if $|X'(u)| \neq 0$.

(also called "regular curves").

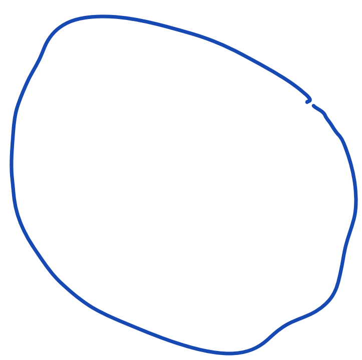
→ X is called an embedded curve if X is a homeomorphism onto its image.

Closed curves would mean immersions X defined on all of $\mathbb{R}$ but which are periodic, i.e., $\exists$ $a > 0$ s.t. $X(u+a) = X(u)$ $\forall$ $u \in I$.

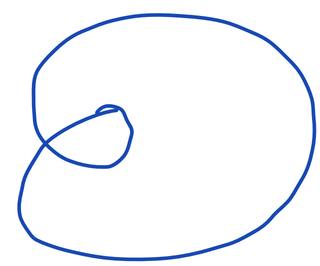
Closed curves w/ an embedding would mean X is injective modulo its periodicity

i.e., $X(u) = X(u') \implies u' - u = ka, \ k \in \mathbb{Z}$.

In this case X is a simple closed curve.



simple closed curve



non-simple closed curve.

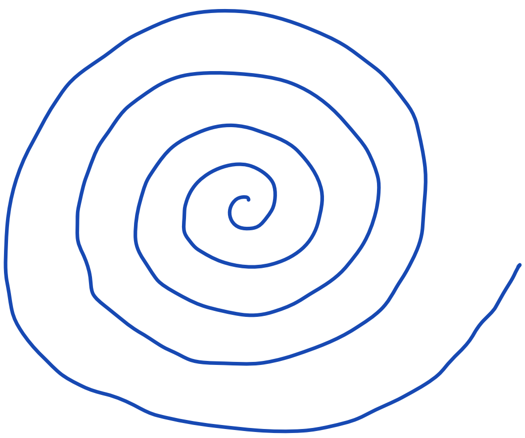
Arc length

Defn :- The arc-length of a curve X starting at some point $X(u_0)$ is the function $s(u)$ w/

$$s(u) = \int_{u_0}^{u} |X'(t)| \, dt.$$

so $s(u_0) = 0$ and it could be positive or negative.

exe. Calculate the arc-length parameter $s(u)$ for
$X(u) = (e^u \cos u, e^u \sin u)$



logarithmic spiral.

If s is the arc length of X starting at $X(u_0)$ then

$$\frac{ds}{du} = \frac{d}{du} \int_{u_0}^{u} |X'(t)|\,dt = |X'(u)|$$

We define the unit tangent vector T to be

$$T = \frac{dX}{ds} = \frac{dX}{du} \cdot \frac{du}{ds} = \frac{X'(u)}{|X'(u)|} \cdot$$

Note:- If v is a unit vector that is a smooth function of u then $v' \perp v$.

In particular, X'' is either zero or perpendicular to X'.

Curvature

Two guiding principles:-

1) the curvature of a curve should be unchanged when the curve is reparametrized.

2) the curvature of a straight line $\equiv 0$ and curvature of bigger circles should be smaller than curvature of smaller circles.

Defn:- If s is the arc-length parameter then the curvature $\kappa(s)$ at $X(s)$ is $\left\| \frac{d^2 X}{ds^2} \right\|$.

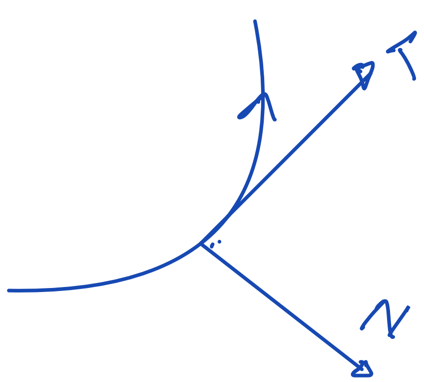
Exercise:- compute the curvature of the helix

$$X(\theta) = (a\cos\theta,\ a\sin\theta,\ b\theta),\ -\infty < \theta < \infty.$$

Let's specialize to $X: (\alpha, \beta) \longrightarrow \mathbb{R}^2$, i.e., to planar curves.

Define the unit normal vector to the curve as

– the unit vector obtained by rotating T counterclockwise by $\pi/2$.



i.e. Our convention is that simple closed curve w/ a counterclockwise parametrisation has an outward pointing normal.

$\because$ T is a unit vector $\Rightarrow$ $\frac{dT}{ds}$ is perpendicular to T and hence parallel to N. Thus $\exists$ some number κ s.t.

$$\left.\begin{aligned} \frac{dT}{ds} &= -\kappa N \\ \frac{dN}{ds} &= \kappa T \end{aligned}\right\} \text{Frenet-Serret equations}$$

Reversing the direction of parametrisation reverses s, T and N $\Rightarrow$ it reverses the sign of κ, but κN is uneffected. κN is called the curvature vector of X.

Also, $\kappa = \left\langle \frac{dN}{ds}, T \right\rangle = -\left\langle \frac{dT}{ds}, N \right\rangle$.

Example:- Consider the round circle of radius r w/ $X(\theta) = (r\cos\theta, r\sin\theta)$.

The arc length parameter $s = \theta r$ and so the arc-length reparametrisation of the curve is

$$X(s) = r\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right)\right).$$

$$X'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right)$$

and $|X'(s)| = 1$

$$X''(s) = \left(-\frac{1}{r}\cos\frac{s}{r}, -\frac{1}{r}\sin\frac{s}{r}\right)$$

$\Rightarrow$ $\|X''(s)\| = \frac{1}{R}$ which is the curvature of the circle.